AIAA JOURNAL
Vol. 33, No. 1, January 1995

Improved Stiffness-Based First-Order Approximations for Structural Optimization

Uri Kirsch*
Technion—Israel Institute of Technology, Haifa 32000, Israel

Improved first-order approximations of displacements, stresses, and forces are presented. The main objectives in developing the method presented are 1) to preserve the ease of implementation and the efficiency of the common first-order approximations and 2) to improve significantly the quality of the results, such that the method can be used in problems with very large changes in the design variables, including geometrical changes and elimination of members. The method is based on results of a single exact analysis and can be used with a general finite element system. It is suitable for different types of design variables and structures. Results obtained by the proposed method are compared with various first-order approximations for modifications in the cross section as well as the geometry and the topology of the structure. It is shown that the proposed approximations are most effective in terms of the accuracy, the efficiency, and the ease of implementation.

Introduction

ONE of the main obstacles in the optimization of structural systems is the high computational cost involved in the solution of large-scale problems. Application of approximation concepts in structural optimization, intended to reduce the computational cost, has been motivated by the following characteristics of practical design problems.

1) The problem size (number of variables and constraints) is usually large. Each element involves at least one variable, and various failure modes under each of several load conditions must be considered.

2) Each redesign involves extensive calculations. The constraints are usually implicit functions of the design variables and evaluation of the constraints value for any assumed design requires the solution of simultaneous equations. In addition, calculation of the constraint derivatives with respect to design variables is often needed.

3) The number of redesigns is large. In general, the solution of optimal design problems is iterative and consists of repeated analyses followed by redesign steps. The number of redesigns and repeated analyses is often a function of the problem dimensionality.

The analysis task will require most of the computational effort; therefore, only methods which do not involve many implicit analyses are suitable for design optimization. Approximate models of the structural behavior reduce the number of exact analyses during the solution process and might affect the overall computational cost more than the choice of the optimization method. Reduction of the computational cost, in turn, allows the solution of practical design problems.[1,2] In general, the following factors are considered in choosing an approximate behavior model for a specific optimal design problem: 1) the accuracy of the calculations, or the quality of the approximations; 2) the computational effort involved, or the efficiency of the method; and 3) the ease of implementation. The implementation effort is weighted against the performance of the algorithms as reflected in their computational efficiency and accuracy.

The various approximations can be divided into the following classes.[1,2]

1) Global approximations (called also multipoint approximations), such as polynomial fitting or reduced basis methods[2–4]: These approximations are obtained by analyzing the structure at a number of design points, and they are valid for the whole design space (or, at least, large regions of it). However, global approximations may require much computational effort in problems with a large number of design variables.

Received Dec. 2, 1993; revision received June 23, 1994; accepted for publication June 27, 1994.

*Professor, Department of Civil Engineering. Member AIAA.

2) Local approximations (called also single-point approximations), such as the first-order Taylor series expansion or the binomial series expansion about a given point in the design space: Local approximations are based on information calculated at a single point. These methods are most efficient, but they are effective only in cases of small changes in the design variables. For large changes in the design the accuracy of the approximations often deteriorates, and they may become meaningless. That is, the approximations are only valid in the vicinity of a design point. To improve the quality of the results, reciprocal cross-sectional areas are often assumed as design variables.[5,6] A hybrid form of the direct and reciprocal approximations which is more conservative than either can also be introduced.[7] This approximation has the advantage of being convex[8]; but it has been found that the hybrid approximation tends to be less accurate than either the direct or the reciprocal approximation. More accurate convex approximations can be introduced by the method of moving asymptotes,[9] but the quality of the results is highly dependent on the selection of the moving asymptotes. Another possibility to improve the quality of the results is to consider second-order approximations,[10,11] but this might increase considerably the computational effort.

3) Combined approximations, which attempt to give global qualities to local approximations: One approach to introduce combined approximations is to scale the initial stiffness matrix such that the changes in the design variables are reduced.[12,13] The advantage of this approach is that, similar to local approximations, the solution is based on results of a single exact analysis. It has been shown that the scaling procedure is useful for various types of design variables and behavior functions. In particular, simplified approximations of homogeneous displacement and stress functions can be achieved.[14] Several criteria for selecting the scaling multiplier have been proposed.[12–14] The concept of scaling has been extended recently to include also the approximate displacements, in addition to the initial stiffness matrix, thereby allowing improved results.[15,16]

The effectiveness of combined approximations of displacements in problems of cross-sectional optimization as well as geometrical and topological optimization has been demonstrated recently.[2,15–17] It has been shown that high quality results can be achieved for very large changes in the design variables. However, the approximations presented in the past are based on second-order information, therefore, the computational effort is somewhat higher than that involved in the common first-order approximations. In addition, approximations of stresses and forces have not been demonstrated.

In this paper improved first-order approximations of displacements, stresses, and forces are introduced. The main objectives in developing the method presented are to preserve the ease of implementation and the efficiency of the common first-order Taylor series approximations and to improve significantly the quality of the results, such that the method can be used in problems with very large changes in the design variables.

The method is based on results of a single exact analysis and can be used with a general finite element system. It is suitable for different types of design variables (e.g., cross-sectional variables, geometrical variables, shape variables) and structures (e.g., trusses, frames, grillages, plates). Results obtained by the proposed combined approximations of order 1 (CA1) are compared with the following methods: 1) the common direct approximations of order 1 (DA1), 2) the common reciprocal approximations of order 1 (RA1), 3) the previously developed combined approximations of order 2 (CA2), and 4) improved reciprocal approximations, called transformed approximations of order 1 (TA1) and exponential approximations of order 1 (EA1).

It will be shown that that the proposed CA1 are most effective in terms of the accuracy, the efficiency, and the ease of implementation. The method provides high quality approximations of the structural behavior for very large changes in the design variables, including changes in the structural topology (elimination of some members and joints).

Common First-Order Approximations

Considering the displacement method analysis equations, the displacements $\boldsymbol{r}$ are computed by the equilibrium equations

$$\boldsymbol{K}\boldsymbol{r} = \boldsymbol{R} \tag{1}$$

where $\boldsymbol{K}$ is the stiffness matrix and $\boldsymbol{R}$ the load vector. Once the displacements are determined, the stresses $\boldsymbol{\sigma}$ and the forces $\boldsymbol{N}$ can readily be calculated by the explicit stress-displacement and force-displacement relations

$$\boldsymbol{\sigma} = \boldsymbol{S}\boldsymbol{r} \tag{2a}$$

$$\boldsymbol{N} = \boldsymbol{W}\boldsymbol{\sigma} = \boldsymbol{W}\boldsymbol{S}\boldsymbol{r} = \boldsymbol{T}\boldsymbol{r} \tag{2b}$$

in which $\boldsymbol{S}$ is the stress-transformation matrix, $\boldsymbol{W}$ is a diagonal matrix giving the force-stress ratios, and matrix $\boldsymbol{T}$ is defined as

$$\boldsymbol{T} = \boldsymbol{W}\boldsymbol{S} \tag{3}$$

The elements of matrices $\boldsymbol{K}$ and $\boldsymbol{W}$ are usually some explicit functions of the design variables, whereas the elements of $\boldsymbol{R}$ and $\boldsymbol{S}$ are often constant.

Taylor Series Expansion

The first-order Taylor series is perhaps the most commonly used approximation in structural optimization. Considering the first-order terms of the displacements, the stresses and the forces, respectively, the resulting DA1 are given by

$$\boldsymbol{r}_D = \boldsymbol{r}^* + \sum_{i=1}^{n} \frac{\partial \boldsymbol{r}^*}{\partial X_i}(X_i - X_i^*) \tag{4a}$$

$$\boldsymbol{\sigma}_D = \boldsymbol{\sigma}^* + \sum_{i=1}^{n} \frac{\partial \boldsymbol{\sigma}^*}{\partial X_i}(X_i - X_i^*) \tag{4b}$$

$$\boldsymbol{N}_D = \boldsymbol{N}^* + \sum_{i=1}^{n} \frac{\partial \boldsymbol{N}^*}{\partial X_i}(X_i - X_i^*) \tag{4c}$$

in which X_i are the original design variables, n is the number of design variables, and asterisks denote values calculated at the initial design $\boldsymbol{X}^*$. It has been shown[18] that approximations of forces [Eqs. (4c)] are of higher quality than approximations of stresses [Eqs. (4b)] for stress constraints. This result also will be demonstrated later in this paper.

A common approach to improve the quality of the results is to assume intermediate variables. Although it might be difficult to select appropriate variables in cases of a general optimization problem where geometrical, topological or shape design variables are considered, the selection of the reciprocal variables Y_i, defined as $Y_i = 1/X_i$, often significantly improves the quality of the results. The resulting first-order expressions RA1, are given by

$$\boldsymbol{r}_R = \boldsymbol{r}^* + \sum_{i=1}^{n} \frac{\partial \boldsymbol{r}^*}{\partial Y_i}(Y_i - Y_i^*) = \boldsymbol{r}^* + \sum_{i=1}^{n} y_i \frac{\partial \boldsymbol{r}^*}{\partial X_i}(X_i - X_i^*) \tag{5a}$$

$$\boldsymbol{\sigma}_R = \boldsymbol{\sigma}^* + \sum_{i=1}^{n} \frac{\partial \boldsymbol{\sigma}^*}{\partial Y_i}(Y_i - Y_i^*) = \boldsymbol{\sigma}^* + \sum_{i=1}^{n} y_i \frac{\partial \boldsymbol{\sigma}^*}{\partial X_i}(X_i - X_i^*) \tag{5b}$$

$$\boldsymbol{N}_R = \boldsymbol{N}^* + \sum_{i=1}^{n} \frac{\partial \boldsymbol{N}^*}{\partial Y_i}(Y_i - Y_i^*) = \boldsymbol{N}^* + \sum_{i=1}^{n} y_i \frac{\partial \boldsymbol{N}^*}{\partial X_i}(X_i - X_i^*) \tag{5c}$$

where the multipliers $y_i = X_i^*/X_i$ can be viewed as correction multipliers of the linear approximations, converting them into nonlinear ones.

It is instructive to note that the RA1 are not suitable for problems where certain X_i approach zero, since the corresponding y_i approach ∞. This is the case, for example, in topological optimization where some cross sections of eliminated members become zero. In addition, approximations about an initial value of zero require some modifications. To illustrate this situation, assume the common case of optimization along the line

$$\boldsymbol{X} = \boldsymbol{X}^* + \alpha\, \Delta\boldsymbol{X}^* \tag{6}$$

where $\boldsymbol{X}^*$ is a given design, α a step size variable, and $\Delta\boldsymbol{X}^*$ a given direction vector. Considering displacement approximations, the DA1 and the RA1 [Eqs. (4a) and (5a), respectively] become

$$\boldsymbol{r}_D = \boldsymbol{r}^* + \frac{\partial \boldsymbol{r}^*}{\partial \alpha}(\alpha - \alpha^*) \tag{7a}$$

$$\boldsymbol{r}_R = \boldsymbol{r}^* + \frac{\alpha^*}{\alpha}\frac{\partial \boldsymbol{r}^*}{\partial \alpha}(\alpha - \alpha^*) \tag{7b}$$

Since $\alpha^* = 0$, Eq. (7b) cannot be used, and modification of the approximate expression is necessary. One possibility is to assume $\alpha^* = \delta\alpha$ (where $\delta\alpha$ is a small number), but it has been found that in this case the results are highly dependent on the selection of $\delta\alpha$.

It should be noted that effective implementation of the RA1 requires calculation of derivatives with respect to all variables, even in cases of a single independent variable α. Assuming the case where the derivative vectors $\partial\boldsymbol{r}/\partial X_i$ are not known from previous calculations, then the common DA1 or the CA1 proposed in this paper involve evaluation of only the single derivative vector $\partial\boldsymbol{r}/\partial\alpha$ [Eq. (7a)]. On the other hand, to exploit the advantage of using the correction multipliers $y_i = X_i^*/X_i$ in the RA1 for the displacements it is necessary to calculate all of the derivative vectors $\partial\boldsymbol{r}/\partial X_i$.

One reason for the better quality of the RA1 is that displacements and stresses for statically determinate structures are often linear functions of the reciprocal variables. For statically indeterminate structures, the use of these variables still proves to be a useful device to obtain better approximations. The advantage of the reciprocal variables can be best understood in cases where the displacements, the stresses, and the forces are homogeneous functions of the design variables. For illustrative purposes, assume the common case of truss cross-sectional variables. The displacements and the stresses are homogeneous functions of degree -1 in the design variables and Eqs. (5) are reduced to the simplified form[2,5]

$$\boldsymbol{r}_R = \sum_{i=1}^{n} \frac{\partial \boldsymbol{r}^*}{\partial Y_i} Y_i \tag{8a}$$

$$\boldsymbol{\sigma}_R = \sum_{i=1}^{n} \frac{\partial \boldsymbol{\sigma}^*}{\partial Y_i} Y_i \tag{8b}$$

It can be observed that for statically determinate structures the displacement and stress derivatives in this case are constant, and Eqs. (8) are exact.

In the examples presented later in this paper, the conservative-convex approximations[7,8] are not considered since it has been found that they tend to be less accurate than either the direct or the reciprocal approximation. Other methods intended to improve the RA1, but highly dependent on some predetermined parameters, will be discussed later.

Binomial Series Expansion

Express the modified equilibrium equations (1), after a change in the initial design, as

$$\boldsymbol{K}\boldsymbol{r} = (\boldsymbol{K}^* + \Delta\boldsymbol{K})(\boldsymbol{r}^* + \Delta\boldsymbol{r}) = \boldsymbol{R} \tag{9}$$

in which $\Delta\boldsymbol{K}$ and $\Delta\boldsymbol{r}$ are the changes in the initial values $\boldsymbol{K}^*$ and $\boldsymbol{r}^*$, respectively. Neglecting the second-order term $\Delta\boldsymbol{K}\,\Delta\boldsymbol{r}$, defining matrix $\boldsymbol{B}$ as

$$\boldsymbol{B} \equiv \boldsymbol{K}^{*-1}\Delta\boldsymbol{K} \tag{10}$$

substituting in Eq. (9) and rearranging yields

$$\boldsymbol{r} = (\boldsymbol{I} - \boldsymbol{B})\boldsymbol{r}^* \tag{11}$$

It has been shown[2] that Eq. (11) is equivalent to the first-order Taylor series expansion for homogeneous displacement functions. Furthermore, this equation is identical to the first-order approximation of the binomial series expansion

$$\boldsymbol{r} = (\boldsymbol{I} - \boldsymbol{B} + \boldsymbol{B}^2 - \cdots)\boldsymbol{r}^* \tag{12}$$

It should be noted that calculation of the binomial series elements involves only forward and backward substitutions if $\boldsymbol{K}^*$ is given in a decomposed form from the initial analysis.[2] In addition, calculation of the binomial series terms does not involve calculation of derivatives.

The first-order approximations presented in this paper will be compared later with second-order approximations evaluated by Eq. (12).

Improved Approximations

Proposed Method: Combined First-Order Approximations

For simplicity of presentation, denote the terms in the DA1 [Eq. (4)] as

$$\boldsymbol{r}_D = \boldsymbol{r}_1 + \boldsymbol{r}_2 \tag{13a}$$

$$\boldsymbol{\sigma}_D = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 \tag{13b}$$

$$\boldsymbol{N}_D = \boldsymbol{N}_1 + \boldsymbol{N}_2 \tag{13c}$$

where subscripts 1 and 2 denote the first and the second terms, respectively, given by

$$\boldsymbol{r}_1 = \boldsymbol{r}^* \qquad \boldsymbol{r}_2 = \sum_{i=1}^{n} \frac{\partial \boldsymbol{r}^*}{\partial X_i}(X_i - X_i^*) \tag{14a}$$

$$\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}^* \qquad \boldsymbol{\sigma}_2 = \sum_{i=1}^{n} \frac{\partial \boldsymbol{\sigma}^*}{\partial X_i}(X_i - X_i^*) \tag{14b}$$

$$\boldsymbol{N}_1 = \boldsymbol{N}^* \qquad \boldsymbol{N}_2 = \sum_{i=1}^{n} \frac{\partial \boldsymbol{N}^*}{\partial X_i}(X_i - X_i^*) \tag{14c}$$

To improve the quality of the approximations, we use the first-order displacement terms as basis vectors in the following reduced basis expression

$$\boldsymbol{r}_I = y_1\boldsymbol{r}_1 + y_2\boldsymbol{r}_2 = \boldsymbol{r}_B\boldsymbol{y} \tag{15}$$

where matrix $\boldsymbol{r}_B$ and the vector $\boldsymbol{y}$ of coefficients to be determined are defined as

$$\boldsymbol{r}_B = \{\boldsymbol{r}_1, \boldsymbol{r}_2\} \qquad \boldsymbol{y}^T = \{y_1, y_2\} \tag{16}$$

Substituting Eqs. (15) into the modified analysis Eqs. (1) and premultiplying by $\boldsymbol{r}_B^T$ yields

$$\boldsymbol{r}_B^T\boldsymbol{K}\boldsymbol{r}_B\,\boldsymbol{y} = \boldsymbol{r}_B^T\boldsymbol{R} \tag{17}$$

Introducing the notation

$$\boldsymbol{K}_R = \boldsymbol{r}_B^T\boldsymbol{K}\boldsymbol{r}_B \qquad \boldsymbol{R}_R = \boldsymbol{r}_B^T\boldsymbol{R} \tag{18}$$

and substituting Eqs. (18) into Eq. (17), we obtain the set of (2×2) equations

$$\boldsymbol{K}_R\,\boldsymbol{y} = \boldsymbol{R}_R \tag{19}$$

Thus, the displacement vector is evaluated by solving first the (2×2) system in Eq. (19) for $\boldsymbol{y}$. The improved approximate displacements $\boldsymbol{r}_I$ are then computed for the given $\boldsymbol{y}$ by Eq. (15).

The corresponding stresses and forces are evaluated by substituting Eqs. (15) into Eqs. (2), giving

$$\begin{aligned}\boldsymbol{\sigma}_I &= \boldsymbol{S}\boldsymbol{r}_I = \boldsymbol{S}(y_1\boldsymbol{r}_1 + y_2\boldsymbol{r}_2)\\ \boldsymbol{N}_I &= \boldsymbol{T}\boldsymbol{r}_I = \boldsymbol{T}(y_1\boldsymbol{r}_1 + y_2\boldsymbol{r}_2)\end{aligned} \tag{20}$$

Denoting

$$\begin{aligned}\boldsymbol{\sigma}_1 &= \boldsymbol{S}\boldsymbol{r}_1 \qquad \boldsymbol{\sigma}_2 = \boldsymbol{S}\boldsymbol{r}_2\\ \boldsymbol{N}_1 &= \boldsymbol{T}\boldsymbol{r}_1 \qquad \boldsymbol{N}_2 = \boldsymbol{T}\boldsymbol{r}_2\end{aligned} \tag{21}$$

and substituting Eqs. (21) into Eqs. (20), then the stress and the force approximations become

$$\begin{aligned}\boldsymbol{\sigma}_I &= y_1\boldsymbol{\sigma}_1 + y_2\boldsymbol{\sigma}_2\\ \boldsymbol{N}_I &= y_1\boldsymbol{N}_1 + y_2\boldsymbol{N}_2\end{aligned} \tag{22}$$

In summary, given the initial stiffness matrix ($\boldsymbol{K}^*$) and the first-order Taylor series terms ($\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{N}_1, \boldsymbol{N}_2$), the proposed CA1 involve the following additional calculations.

1) The modified matrix $\boldsymbol{K} = \boldsymbol{K}^* + \Delta\boldsymbol{K}$ is introduced.

2) The reduced matrix $\boldsymbol{K}_R$ and the reduced vector $\boldsymbol{R}_R$ are calculated by Eqs. (18).

3) The coefficients $\boldsymbol{y}$ are calculated by solving the set of (2×2) Eq. (19).

4) The final displacements, stresses, and forces are evaluated by Eqs. (15) and (22).

It will be shown later by several numerical examples that the proposed procedure improves significantly the quality of the results. The multipliers y_1 and y_2 can be viewed as scaling parameters, intended to improve the first-order Taylor series approximations. The following special cases, for selecting the $\boldsymbol{y}$ multipliers, can readily be identified: 1) scaling of the initial behavior (displacements, stresses and forces), obtained for some $y_1 = y$ and $y_2 = 0$; 2) scaling of the modified approximate behavior, obtained for some $y_1 = y_2 = y$; and 3) the common direct approximations DA1 [Eq. (4)], obtained for $y_1 = y_2 = 1$.

It should be noted that the approximations presented are often valid for the whole design space. Therefore, it is necessary to evaluate all displacement degrees of freedom and not only the constrained degrees of freedom, since the latter might change from one design point to another. In cases where it is necessary to evaluate only a limited number of displacements, it is possible to introduce a matrix $\boldsymbol{W}$ of relative weights for the various degrees of freedom, as presented elsewhere.[15] Then, if only several displacements are of interest, the corresponding elements of $\boldsymbol{W}$ can be selected accordingly.

The method presented can be used also to evaluate effectively constraint derivatives (sensitivity analysis coefficients), as demonstrated recently.[19] As a result, the number of exact analyses and the total computational cost involved in the solution process are further reduced.

Combined Second-Order Approximations

For purposes of comparison, results for CA2 also will be presented. Although the second-order terms of the Taylor series can be used for this purpose, it has been found that the binomial series terms are easy to implement and provide high quality results.[2,15–17] Considering the three basis vectors of the binomial series (12)

$$\begin{aligned} r_1 &= r^* \\ r_2 &= -Br^* \\ r_3 &= B^2 r^* \end{aligned} \tag{23}$$

then the reduced basis expression (15) becomes

$$r_I = y_1 r_1 + y_2 r_2 + y_3 r_3 = r_B\, y \tag{24}$$

where

$$\begin{aligned} r_B &= \{r_1, r_2, r_3\} \\ y^T &= \{y_1, y_2, y_3\} \end{aligned} \tag{25}$$

The resulting stresses and forces are [see Eqs. (22)]

$$\begin{aligned} \sigma_I &= y_1\sigma_1 + y_2\sigma_2 + y_3\sigma_3 \\ N_I &= y_1 N_1 + y_2 N_2 + y_3 N_3 \end{aligned} \tag{26}$$

in which

$$\begin{aligned} \sigma_1 &= Sr_1 \qquad \sigma_2 = Sr_2 \qquad \sigma_3 = Sr_3 \\ N_1 &= Tr_1 \qquad N_2 = Tr_2 \qquad N_3 = Tr_3 \end{aligned} \tag{27}$$

Evidently, the CA2 [Eqs. (24) and (26)] involve some more calculations than the proposed CA1. Specifically, the additional second-order terms r_3, σ_3, and N_3 are first calculated, and the set of (3×3) Eq. (19) is then solved for y. However, it has been noted[2,15] that the amount of these extra calculations is not significant. In addition, the quality of the results is better than that of the first-order approximations. It has been shown also that the CA2 is a powerful tool for effective evaluation of the structural behavior and the sensitivity analysis vectors in optimization of the cross sections, as well as the geometry and the topology of the structure.[15,17,19]

Improved Reciprocal Approximations

It has been noted earlier that the RA1 [Eqs. (5)] are not suitable for problems where some cross sections approach zero. To overcome this difficulty, it is possible to use the transformation[20]

$$Y_i = \frac{1}{X_i + \delta X_i} \tag{28}$$

where the values of δX_i are typically small compared to representative values of the corresponding X_i. Using Eqs. (5), the modified multipliers y_i are given by

$$y_i = (X_i^* + \delta X_i)/(X_i + \delta X_i) \tag{29}$$

Thus, Eq. (29) is used instead of the common reciprocal multipliers $y_i = X_i^*/X_i$. The main problem in using these TA1 is that the quality of the results is highly dependent on the selected δX_i. It will be shown subsequently by some examples that adequate approximations might be obtained only for certain δX_i values.

An alternative approach to improve the quality of the RA1 results is to modify the multipliers $y_i = X_i^*/X_i$ in Eqs. (5) to obtain

$$y_i = (X_i^*/X_i)^m \tag{30}$$

where m is a parameter to be selected. The resulting EA1 might significantly improve the quality of the approximations, as will be shown subsequently. However, similar to the TA1, the results are highly dependent on the selected m, and it is not always an easy task to find an appropriate value. In the common RA1 the selected m is 1; some considerations for evaluating m are discussed elsewhere.[14,21]

In summary, the various y_i multipliers discussed in this paper are given in Table 1.

Numerical Examples

Cross-Sectional Optimization

To illustrate the quality of the approximations presented, consider the well-documented 10-bar truss shown in Fig. 1. The truss is subjected to a single loading condition of two concentrated loads, and the design variables are the 10 cross-sectional areas X_i $(i = 1, \ldots, 10)$. The modulus of elasticity is $E = 30{,}000$ and the eight analysis unknowns are the horizontal (to the right) and the vertical (downward) displacements at joints 1, 2, 3, and 4, respectively. The initial cross-sectional areas are $X = \mathbf{1.0}$, the stress constraints are $-\mathbf{25.0} \le \sigma \le \mathbf{25.0}$, and the minimum size constraints are $\mathbf{0.001} \le X$. Assuming weight as an objective function, the optimal design is[17]

$$X_{\rm opt}^T = \{8.0, 0.001, 8.0, 4.0, 0.001, 0.001, 5.66, 5.66, 5.66, 0.001\}$$

The percentages of change in the design variables during the solution process are as follows: variable 1, 700%; 2, −99.9%; 3, 700%; 4, 300%; 5, −99.9%; 6, −99.9%; 7, 46.6%; 8, 46.6%; 9, 46.6%; and 10, −99.9%.

That is, the cross sections of members 1, 3, 4, and 7–9, are increased and—at the same time—the topology is practically changed by eliminating members 2, 5, 6, 10, and joint 2 (displacements 3 and 4).

Solving the optimization problem using the proposed CA1, the final optimum is achieved after only two exact analyses. This result, which is significantly better than the number of analyses required for the DA1 and RA1 reported in the literature, shows the potential of the method presented to provide effective approximations for most of the design space. In addition, the method is used to evaluate both the constraint values and the constraint derivatives by an algorithm presented recently.[19]

To demonstrate the quality of the results for very large changes in the design variables, assume the line from the initial design to the optimal design, given by

$$X = X^* + \alpha \Delta X^*$$

where α is the step size variable and ΔX^* is defined as

$$\Delta X^{*T} = \{7.0, -0.999, 7.0, 3.0, -0.999, -0.999, 4.66, 4.66, 4.66, -0.999\}$$

Approximate results obtained by CA1, CA2, RA1, and DA1 have been compared with the exact solution of this example. Approximate displacements, stresses and members' forces for various α values are shown in Tables 2–4 and in Fig. 2. The following observations have been made.

1) As expected, the errors in all of the listed approximations increase with α. The results achieved by the proposed CA1 are significantly better than those obtained by either the DA1 or the RA1, but the best results have been achieved by the CA2. As mentioned earlier, the latter approximations are presented only for purposes of comparison but, despite the larger computational effort, the CA2 are highly recommended in cases where higher quality of the approximations is needed.

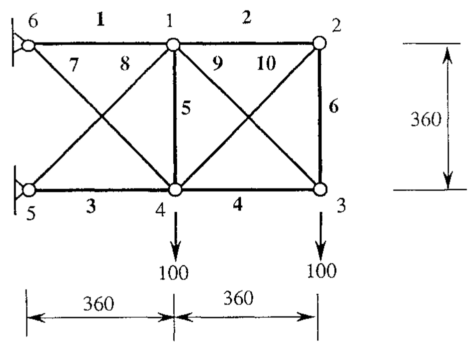

Fig. 1 A 10-bar truss.

Table 1 Multipliers y_i for various approximations

Method	y_i		
DA1	$y_1 = y_2 = 1.0$		
RA1	$y_1 = 1.0$	$y_{i+1} = X_i^*/X_i$	$(i = 1, \ldots, n)$[a]
EA1	$y_1 = 1.0$	$y_{i+1} = (X_i^*/X_i)^m$	$(i = 1, \ldots, n)$
TA1	$y_1 = 1.0$	$y_{i+1} = (X_i^* + \delta X_i)/(X_i + \delta X_i)$	$(i = 1, \ldots, n)$
CA1	$\{y_1, y_2\}^T = \left(\boldsymbol{r}_B^T \boldsymbol{K} \boldsymbol{r}_B\right)^{-1} \boldsymbol{r}_B^T \boldsymbol{R}$		
CA2	$\{y_1, y_2, y_3\}^T = \left(\boldsymbol{r}_B^T \boldsymbol{K} \boldsymbol{r}_B\right)^{-1} \boldsymbol{r}_B^T \boldsymbol{R}$		

[a]Number of variables is indicated by n.

Table 2 Various approximations of displacements, cross-section changes

		Displacements							
α	Method	1	2	3	4	5	6	7	8
0.4	Exact	0.61	1.73	0.89	4.21	−1.12	4.49	−0.65	1.90
	CA2	0.61	1.71	0.88	4.20	−1.12	4.47	−0.65	1.91
	CA1	0.59	1.78	0.84	4.14	−1.04	4.38	−0.64	1.98
	RA1	0.59	1.66	1.03	4.92	−1.36	5.35	−0.68	1.98
0.8	Exact	0.36	1.06	0.56	2.61	−0.71	2.82	−0.37	1.10
	CA2	0.35	1.01	0.54	2.58	−0.71	2.77	−0.38	1.14
	CA1	0.33	1.08	0.50	2.51	−0.63	2.67	−0.36	1.20
	RA1	0.26	0.75	1.33	6.2	−2.01	7.27	−0.47	1.43
1.0	Exact	0.30	0.90	0.49	2.21	−0.60	2.40	−0.30	0.90
	CA2	0.29	0.84	0.45	2.17	−0.61	2.34	−0.31	0.95
	CA1	0.28	0.89	0.41	2.10	−0.53	2.24	−0.30	1.01
	RA1	a	a	a	a	a	a	a	a

[a]Meaningless results.

Table 3 Various approximations of stresses, cross-section changes

		Stresses									
α	Method	1	2	3	4	5	6	7	8	9	10
0.4	Exact	51.2	23.0	−54.1	−39.2	13.7	23.0	52.0	−46.6	42.5	−32.4
	CA2	50.9	22.3	−54.4	−39.0	16.7	22.3	52.5	−45.9	42.8	−31.6
	CA1	49.4	20.2	−53.0	−33.5	16.6	20.2	56.0	−49.5	40.3	−28.6
	RA1	49.4	36.6	−55.7	−57.7	26.8	36.6	54.7	−44.4	72.6	−51.4
	DA1	a	a	a	a	a	a	a	a	a	a
0.8	Exact	29.9	17.0	−30.7	−28.4	3.9	17.0	30.6	−29.1	28.8	−24.1
	CA2	29.3	15.6	−31.4	−28.0	11.0	15.6	31.9	−27.5	29.0	−22.0
	CA1	28.0	13.3	−30.2	−22.3	10.8	13.3	35.1	−30.8	26.3	−18.8
	RA1	23.0	88.2	−37.5	−130.4	58.3	88.2	41.1	−19.2	176.7	−124.6
	DA1	a	a	a	a	a	a	a	a	a	a

[a]Meaningless results.

Table 4 Various approximations of members' forces, cross-sectional changes

		Forces									
α	Method	1	2	3	4	5	6	7	8	9	10
0.4	Exact	194.5	13.7	−205.5	−86.3	8.2	13.7	149.2	−133.6	122.0	−19.4
	CA2	193.5	13.4	−206.6	−85.8	10.0	13.4	150.6	−131.6	122.6	−19.0
	CA1	188.0	12.1	−201.3	−73.7	9.9	12.1	160.5	−141.8	115.5	−17.1
	DA1	190.7	−8.9	−209.3	−109.0	−18.4	−8.9	154.5	−128.4	154.0	12.8
0.8	Exact	197.4	3.4	−202.6	−96.6	0.8	3.4	145.1	−137.7	136.6	−4.8
	CA2	193.1	3.1	−207.3	−95.3	2.2	3.1	151.3	−130.3	137.4	−4.4
	CA1	184.6	2.7	−199.2	−76.0	2.2	2.7	166.2	−145.9	124.5	−3.8
	DA1	186.1	−57.9	−213.9	−158.2	−72.2	−57.9	161.0	−122.0	223.7	82.3

2) The displacement approximations obtained by the proposed CA1 (Table 2) are reasonable even for $\alpha = 1.0$ (the optimum), where some members are practically eliminated from the structure. On the other hand, the errors obtained by the RA1 for some displacements (3–6) are much larger: 15–20% for $\alpha = 0.2$; 45–60% for $\alpha = 0.4$; 140–180% for $\alpha = 0.8$; and meaningless results for $\alpha = 1.0$.

3) It has been noted previously that effective implementation of the RA1 along a line involves calculation of the derivative vectors with respect to all variables X_i, whereas the proposed CA1 require calculation of only derivatives with respect to the single independent variable α. Displacement approximations of r_6 (the vertical displacement at joint 3) are shown in Fig. 2a. It can be seen that in this case the DA1 are meaningless even for $\alpha = 0.2$; the RA1 are poor for large α values; and the proposed CA1 and the CA2 are the best.

4) The quality of stresses evaluated by the CA1 is relatively good (Table 3 and Fig. 2b). It can be seen that larger errors in stresses are obtained for members with forces approaching zero. Some of the results obtained by the DA1 are meaningless even for $\alpha = 0.2$.

Table 5 Various approximations of displacements, X_2 change (members 2, 5, 6, 10)

		Displacements							
X_2	Method	1	2	3	4	5	6	7	8
0.6	Exact	2.34	5.56	2.92	13.04	−3.32	13.62	−2.46	6.03
	CA2	2.34	5.56	2.92	13.04	−3.32	13.62	−2.46	6.03
	CA1	2.33	5.53	2.97	12.10	−2.97	13.94	−2.47	6.06
	RA1	2.33	5.54	2.95	13.20	−3.37	13.81	−2.47	6.05
	DA1	2.34	5.56	2.89	12.97	−3.21	13.54	−2.46	6.03
	EA1[a]	2.33	5.52	3.00	13.42	−3.46	14.09	−2.47	6.07
0.2	Exact	2.34	5.57	3.08	13.66	−3.51	14.40	−2.46	6.02
	CA2	2.34	5.57	3.08	13.66	−3.51	14.40	−2.46	6.02
	CA1	2.32	5.50	3.04	13.62	−3.53	14.33	−2.47	6.08
	RA1	2.28	5.33	3.55	15.94	−4.37	17.22	−2.52	6.26
	DA1	2.34	5.53	2.97	13.30	−3.18	13.95	−2.47	6.06
	EA1[a]	2.34	5.56	2.90	13.00	−3.30	13.56	−2.46	6.03
0.001	Exact	2.40	5.79	3.35	14.23	−3.60	15.18	−2.40	5.80
	CA2	2.40	5.79	3.35	14.23	−3.60	15.18	−2.40	5.80
	CA1	2.32	5.47	3.13	14.01	−3.67	14.83	−2.48	6.11
	RA1	[b]	[b]	[b]	[b]	[b]	[b]	[b]	[b]
	DA1	2.33	5.52	3.01	13.47	−3.47	14.15	−2.47	6.07
	EA1[a]	2.31	5.46	3.19	14.29	−3.77	15.17	−2.49	6.13

[a]For $m = -1$. [b]Meaningless results.

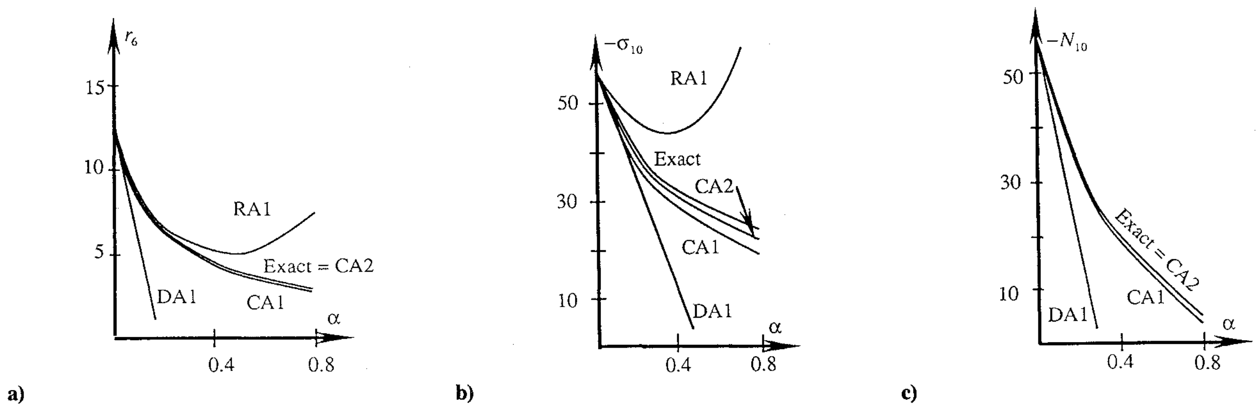

Fig. 2 a) Approximations of the displacement r_6, b) approximations of the stress in the vanishing member 10, and c) approximations of the force in the vanishing member 10.

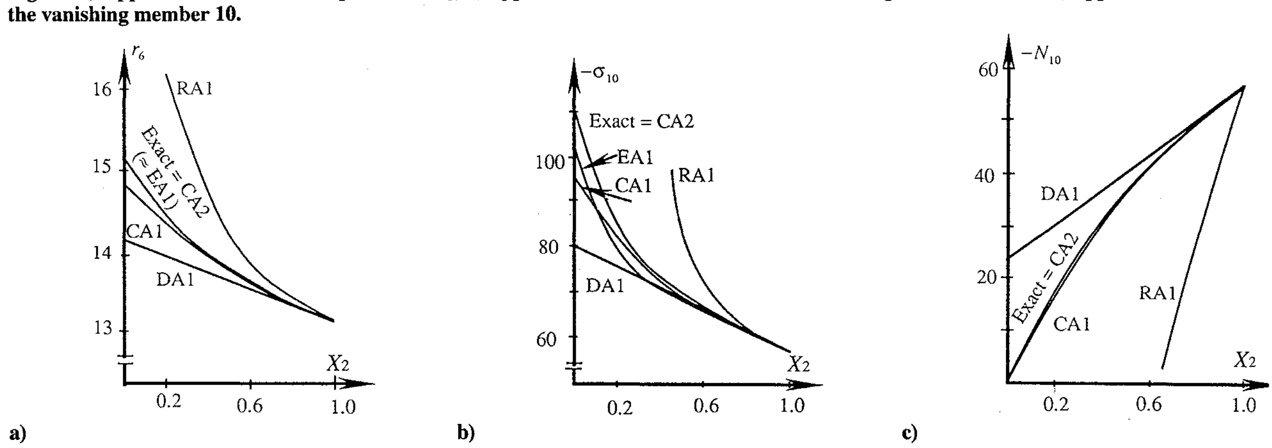

Fig. 3 Approximations in terms of X_2: a) displacement, b) stress, and c) force.

5) Table 4 shows that reasonable results might be obtained for forces by the DA1, as suggested elsewhere.[18] Specifically, the quality of the forces obtained by the DA1 is reasonable for members 1, 3, 7, and 8; larger errors in forces are obtained for members 4, 9, and for members 2, 5, 6, and 10 that are vanishing at the optimum (Fig. 2c).

Elimination of Members

In many structural optimization problems the cross section in some members is reduced during the solution process and might approach zero. Such members are eliminated from the structure, and the initial topology is changed. To illustrate the quality of the approximations in topological optimization, assume only changes in X_2, the cross-sectional area of the eliminated members 2, 5, 6, and 10. Results obtained by the CA2, CA1, RA1, DA1 and EA1 (for $m = -1$) are shown in Tables 5–7, and in Fig. 3. The following observations have been made.

1) The CA2 provides, practically, the exact solution and the CA1 provides very good results.

Table 6 Various approximations of stresses, X_2 change (members 2, 5, 6, 10)

		Stresses									
X_2	Method	1	2	3	4	5	6	7	8	9	10
0.6	Exact	194.8	48.2	−205.2	−71.1	39.6	48.2	148.7	−134.1	100.5	−68.1
	CA2	194.8	48.2	−205.2	−71.1	39.6	48.2	148.7	−134.1	100.5	−68.1
	CA1	194.7	48.1	−205.3	−71.1	39.0	48.0	148.9	−133.9	100.6	−67.8
	RA1	194.4	51.2	−205.6	−75.6	42.5	51.2	149.3	−133.6	107.0	−72.3
	DA1	194.8	46.8	−205.2	−69.2	39.4	46.8	148.7	−134.1	98.0	−66.0
	EA1[a]	194.7	47.1	−205.2	−69.8	39.9	47.1	148.8	−134.1	98.8	−66.5
0.2	Exact	195.1	61.8	−204.9	−87.6	37.4	61.8	148.3	−134.5	124.0	−87.3
	CA2	195.1	61.8	−204.9	−87.6	37.4	61.8	148.3	−134.5	124.0	−87.3
	CA1	193.7	59.3	−206.2	−87.6	48.9	59.3	150.2	−132.5	123.9	−83.8
	RA1	189.8	106.4	−210.2	−154.1	77.5	106.2	155.7	−127.1	218.3	−150.2
	DA1	194.2	53.4	−205.8	−78.7	43.1	53.4	149.5	−133.3	111.4	−75.4
	EA1[a]	194.0	55.7	−206.0	−82.0	45.4	55.7	149.8	−133.1	116.1	−78.7
0.001	Exact	199.9	79.1	−200.1	−99.9	0.6	79.1	141.5	−141.3	141.3	−111.9
	CA2	199.9	79.0	−200.1	−99.9	1.0	79.0	141.5	−141.3	141.3	−112.0
	CA1	193.0	68.0	−206.8	−99.1	53.0	68.0	151.5	−131.6	140.2	−96.0
	RA1	[b]	[b]	[b]	[b]	[b]	[b]	[b]	[b]	[b]	[b]
	DA1	194.0	56.7	−206.0	−83.4	46.0	56.7	149.9	−132.9	118.1	−80.1
	EA1[a]	192.6	73.2	−207.4	−106.8	56.5	73.1	151.8	−131.1	151.3	−103.3

[a]For $m = -1$. [b]Meaningless results.

Table 7 Various approximations of members' forces, X_2 change (members 2, 5, 6, 10)

		Forces									
X_2	Method	1	2	3	4	5	6	7	8	9	10
0.6	Exact	194.8	28.9	−205.2	−71.1	23.7	28.9	148.7	−134.1	100.5	−40.9
	CA2	194.8	28.9	−205.2	−71.1	23.7	28.9	148.7	−134.1	100.5	−40.9
	CA1	194.7	28.8	−205.3	−71.1	24.3	28.8	148.9	−133.9	100.6	−40.7
	RA1	194.4	20.0	−205.6	−75.6	15.7	20.0	149.3	−133.6	107.0	−28.1
	DA1	194.8	30.8	−205.2	−69.2	25.2	30.8	148.7	−134.1	98.0	−43.3
0.2	Exact	195.1	12.4	−204.9	−87.6	7.5	12.4	148.3	−134.5	124.0	−17.5
	CA2	195.1	12.4	−204.9	−87.6	7.5	12.4	148.3	−134.5	124.0	−17.5
	CA1	193.7	11.9	−206.2	−87.6	9.6	11.9	150.2	−132.5	123.9	−16.8
	RA1	189.8	[a]	−210.2	−154.1	[a]	[a]	155.7	−127.1	218.3	[a]
	DA1	194.2	21.3	−205.8	−78.7	14.7	21.3	149.5	−133.3	111.4	−30.0
0.001	Exact	199.9	0.1	−200.1	−99.9	0.0	0.1	141.5	−141.3	141.3	−0.1
	CA2	199.9	0.1	−200.1	−99.9	0.0	0.1	141.5	−141.3	141.3	−0.1
	CA1	193.0	0.1	−206.8	−99.1	0.0	0.1	151.5	−131.6	140.2	−0.1
	RA1	[a]	[a]	[a]	[a]	[a]	[a]	[a]	[a]	[a]	[a]
	DA1	194.0	16.6	−206.0	−83.4	10.5	16.6	149.9	−132.9	118.1	−23.4

[a]Meaningless results.

Table 8 Exact and proposed approximate displacements, changes in geometry

		Displacements							
Y	Method	1	2	3	4	5	6	7	8
540	Exact	1.55	3.94	1.82	7.84	−2.18	8.47	−1.66	4.44
	CA2	1.53	3.93	1.81	7.83	−2.17	8.46	−1.64	4.45
	CA1	1.49	4.02	1.71	7.86	−2.06	8.40	−1.60	4.48
	Error, %	3.4	1.9	6.4	0.2	5.4	0.7	3.0	0.9
720	Exact	1.15	3.67	1.34	6.60	−1.66	7.36	−1.25	4.24
	CA2	1.14	3.67	1.34	6.62	−1.68	7.35	−1.24	4.25
	CA1	1.17	3.78	1.26	6.72	−1.61	7.29	−1.28	4.27
	Error, %	1.0	3.1	6.6	1.7	3.1	1.0	2.3	0.6

2) The RA1 and the DA1 provide poor results and are not suitable for topological optimization.

3) The EA1 and the TA1 can improve the results obtained by the RA1 only for appropriate selection of the parameters m and δX_2. Poor results are obtained for small δX_2, and the approximations tend to the DA1 for large δX_2.

Geometrical Optimization

To demonstrate the quality of the approximations achieved by the proposed CA1 in geometrical optimization, assume the depth of the 10-bar truss Y as a geometrical variable and a uniform cross-sectional area for all members. Approximate displacements and forces evaluated for $Y = 540$ and $Y = 720$ (increase in the depth of the truss by 50% and 100%, respectively) are summarized in Tables 8 and 9. The errors in the displacements, obtained by the CA1 and given in Table 8, are defined as

$$\text{error}[\%] = \left| \frac{r(\text{exact}) - r(\text{CA1})}{r(\text{exact})} \right|$$

It can be observed that very good approximations are achieved for these large changes in the geometry. Specifically, errors of less than 7% in the displacements have been achieved by the proposed CA1.

Table 9 Exact and proposed approximate forces, changes in geometry

		Forces									
Y	Method	1	2	3	4	5	6	7	8	9	10
540	Exact	128.8	23.1	−137.9	−43.6	27.8	34.6	128.4	−112.0	78.6	−41.6
	CA2	128.8	23.1	−137.9	−43.6	27.8	34.6	128.4	−112.0	78.6	−41.6
	CA1	128.8	23.1	−137.9	−43.6	27.8	34.6	128.4	−112.0	78.6	−41.6
720	Exact	96.1	15.9	−103.9	−34.1	23.9	31.7	120.6	−103.0	76.3	−35.5
	CA2	94.7	16.7	−103.3	−36.5	24.1	30.2	121.1	−103.5	75.6	−36.1
	CA1	97.1	7.5	−106.3	−27.6	20.3	23.9	121.0	−106.6	70.8	−39.4

Concluding Remarks

Approximations of the structural behavior in terms of the design variables are essential in optimization of large-scale structures, where the time consuming analysis is repeated many times. Local approximations, such as the common DA1 or RA1, are most efficient, but the quality of the results is often insufficient, particularly in cases of large changes in the design variables. Modification of the RA1 by methods such as the EA1 and the TA1 presented in the paper can significantly improve the results. However, the quality of the approximations is highly dependent on some parameters, and it is not straightforward to find appropriate values for these parameters.

The CA1 presented in this study is based on combining the computed terms of the DA1, used as high quality basis vectors, and coefficients of a reduced basis expression. The latter coefficients can readily be determined by solving a reduced set of (2×2) analysis equations. The advantage is that the efficiency of the DA1 and the improved quality of global approximations are combined to achieve an effective solution procedure.

Similar to the DA1, the CA1 presented is based on results of a single exact analysis and can be used with a general finite element system. It is suitable for different types of structure, such as trusses, frames, grillages, and plates. Although the CA1 can use easy-to-implement basis vectors that do not involve calculation of derivatives (such as the binomial series terms), the common first-order Taylor series terms are assumed in this paper for purposes of comparison. Given the initial stiffness matrix $\boldsymbol{K}^*$ and the DA1 terms, it is then necessary for each trial design only to introduce the modified stiffness matrix $\boldsymbol{K} = \boldsymbol{K}^* + \Delta\boldsymbol{K}$ and to evaluate the displacements, the stresses, and the forces by the simple algebraic operations of Eqs. (18), (19), (15), and (22).

In the examples presented, good approximations of displacements, forces, and stresses have been achieved by the proposed CA1 for very large changes in the design variables. The quality of the approximations achieved by the CA2 is better, at the cost of more computational effort. Computational considerations related to the CA2 are discussed elsewhere.[2,15,16] Generally speaking, it is recommended to use the latter method in cases where the proposed CA1 is not adequate, and higher quality of the approximations is needed.

The results achieved by the proposed CA1 are better than those obtained by the RA1. The RA1 might not be adequate for large changes in the design variables. In addition, effective implementation of the RA1 requires calculation of derivatives with respect to all variables even in cases of a single independent variable (e.g., step-size variable).

In conclusion, the CA1 presented is a powerful tool to achieve efficient and high quality approximations of the constraints in structural optimization problems. It has high potential in future applications in such areas as shape optimization and optimization by genetic algorithms.

Acknowledgment

The author is indebted to "The Fund for the Promotion of Research at the Technion" for supporting this work.

References

[1]Barthelemy, J.-F. M., and Haftka, R. T., "Recent Advances in Approximation Concepts for Optimum Structural Design," Proceedings of NATO/DFG ASI on Optimization of Large Structural Systems, Berchtesgaden, Germany, 1991, pp. 235-256.

[2]Kirsch, U., *Structural Optimizations, Fundamentals and Applications*, Springer-Verlag, Heidelberg, 1993.

[3]Fox, R. L., and Miura, H., "An Approximate Analysis Technique for Design Calculations," *AIAA Journal*, Vol. 9, 1971, pp. 177–179.

[4]Haftka, R. T., Nachlas, J. A., Watson, L. T., Rizzo, T., and Desai, R., "Two-point Constraint Approximation in Structural Optimization," *Computer Methods of Applied Mechanical Engineering*, Vol. 60, 1989, pp. 289–301.

[5]Fuchs, M. B., "Linearized Homogeneous Constraints in Structural Design," *International Journal of Mechanical Science*, Vol. 22, 1980, pp. 333–400.

[6]Schmit, L. A., and Farshi, B., "Some Approximation Concepts for Structural Synthesis," *AIAA Journal*, Vol. 11, 1974, pp. 489–494.

[7]Starnes, J. H., Jr., and Haftka, R. T., "Preliminary Design of Composite Wings for Buckling Stress and Displacement Constraints," *Journal of Aircraft*, Vol. 16, 1979, pp. 564–570.

[8]Fleury, C., and Braibant, V., "Structural Optimization: a New Dual Method Using Mixed Variables," *International Journal of Numerical Methods in Engineering*, Vol. 23, 1986, pp. 409–428.

[9]Svanberg, K., "The Method of Moving Asymptotes—a New Method for Structural Optimization," *International Journal of Numerical Methods in Engineering*, Vol. 24, 1987, pp. 359–373.

[10]Fleury, C., "Efficient Approximation Concepts Using Second Order Information," *International Journal of Numerical Methods in Engineering*, Vol. 28, 1989, pp. 2041–2058.

[11]Fleury, C., "First and Second Order Convex Approximation Strategies in Structural Optimization," *Structural Optimization*, Vol. 1, 1989, pp. 3–10.

[12]Kirsch, U., "Approximate Behavior Models for Optimum Structural Design," *New Directions in Optimum Structural Design*, edited by E. Atrek, R. H. Gallagher, K. M. Ragsdell, and O. C. Zienkiewicz, John Wiley, New York, 1984.

[13]Kirsch, U., and Toledano, G., "Approximate Reanalysis for Modifications of Structural Geometry," *Computers and Structures*, Vol. 16, 1983, pp. 269–279.

[14]Hjali, R. M., and Fuchs, M. B., "Generalized Approximations of Homogeneous Constraints in Optimal Structural Design," *Computer Aided Optimum Design of Structures*, edited by C. A. Brebbia and S. Hernandez, Springer-Verlag, Berlin, 1989, pp. 167–178.

[15]Kirsch, U., "Reduced Basis Approximations of Structural Displacements for Optimal Design," *AIAA Journal*, Vol. 29, 1991, pp. 1751–1758.

[16]Kirsch, U., "Approximate Reanalysis Methods," *Structural Optimization: Status and Promise*, edited by M. P. Kamat, Vol. 150, AIAA, Washington, DC, 1993.

[17]Kirsch, U., "Approximate Reanalysis for Topological Optimization," *Structural Optimization*, Vol. 6, 1993, pp. 143–150.

[18]Vanderplaats, G. N., and Salajegheh, E., "A New Approximation Method for Stress Constraints in Structural Synthesis," *AIAA Journal*, Vol. 27, 1989, pp. 352–358.

[19]Kirsch, U., "Effective Sensitivity Analysis for Structural Optimization," *Computer Methods in Applied Mechanics and Engineering*, Vol. 117, 1994, pp. 143–156.

[20]Haftka, R. T., and Kamat, M. P., *Elements of Structural Optimization*, Martinus Nijhoff, Dordrecht, The Netherlands, 1985.

[21]Fadel, G. M., Riley, M. F., and Barthelemy, J. M., "Two Point Exponential Approximation Method for Structural Optimization," *Structural Optimization*, Vol. 2, 1990, pp. 117–124.